ON THE STABILITY OF DIFFERENCE SCHEMES FOR NONLINEAR HEAT CONDUCTION PROBLEMS

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A nonlinear heat conduction problem with nonlinear boundary conditions is solved by the finite-difference method. The stability of the difference scheme is investigated.

Some questions on the stability of difference schemes for nonlinear heat conduction problems are studied in [7-10]. In contrast to these papers, we investigate here the stability of an implicit two-layered difference scheme with nonlinear boundary conditions.

The questions of the existence and uniqueness of the solution of the problems under consideration, as well as the convergence of the solution of the difference schemes to the solution of the differential equation problem, are studied in [1, 2, 4-6].

1. We consider the following problem:

$$\rho(u)\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left[\lambda(u)\frac{\partial u}{\partial x}\right]-c(x,\ t,\ u)u; \tag{1}$$

$$u(x,\ 0)=\varphi(x);\quad u(0,\ t)=q_1(t);\quad u(l,\ t)=q_2(t); \tag{2}$$

$$\rho>0;\quad c>0;\quad \lambda>0;\quad \left|\frac{dq_1}{dt}\right|\leqslant M;\quad \left|\frac{dq_2}{dt}\right|\leqslant M. \tag{3}$$

The finite-difference scheme

$$\begin{gathered}\rho_{i,n-1}\frac{u_{in}-u_{i,n-1}}{\Delta t}=\frac{\lambda_{i,n-1}(u_{i+1,n}-u_{i,n})-\lambda_{i-1,n-1}(u_{in}-u_{i-1,n})}{\Delta x^2}-c_{i,n-1}u_{in};\\ u_{i0}=\varphi_i;\quad u_{0n}=q_1^{(n)};\quad u_{Nn}=q_2^{(n)}\end{gathered} \tag{4}$$

approximates the problem (1)-(2) with order $o(\Delta t+\Delta x^2)$. We transform (4) into the form

$$\begin{gathered}\lambda_{i-1,n-1}\frac{\Delta t}{\Delta x^2}u_{i-1,n}-\left[\rho_{i,n-1}+\Delta t c_{i,n-1}+(\lambda_{i,n-1}+\lambda_{i-1,n})\frac{\Delta t}{\Delta x^2}\right]u_{in}+\lambda_{i,n-1}\frac{\Delta t}{\Delta x^2}u_{i+1,n}=-\rho_{i,n-1}u_{i,n-1},\\ u_{0,n}=q_1^{(n-1)}+\Delta t\frac{q_1^{(n)}-q_1^{(n-1)}}{\Delta t};\quad u_{Nn}=q_2^{(n-1)}+\Delta t\frac{q_2^{(n)}-q_2^{(n-1)}}{\Delta t};\quad u_{i0}=\varphi_i.\end{gathered} \tag{5}$$

We obtain a linear implicit scheme, where the coefficients of the unknowns are the known values of the functions λ, ρ, and c on the previous layer. We investigate the stability of the scheme (5). We represent the solution of problem (5) in the form

$$u_{in}=W_{in}+V_{in}\Delta t, \tag{6}$$

where W_{in} and V_{in} are the solutions of the problems

$$\begin{gathered}AW_{i-1}-2BW_i+CW_{i+1}=-\rho_{i,n-1}u_{i,n-1},\\ W_0=q_1^{(n-1)},\ W_N=q_2^{(n-1)};\end{gathered} \tag{7}$$

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$$AV_{i-1}-2BV_i+CV_{i+1}=0,\quad V_0=\frac{q_1^{(n)}-q_1^{(n-1)}}{\Delta t},\quad V_N=\frac{q_2^{(n)}-q_2^{(n-1)}}{\Delta t}. \tag{8}$$

Here

$$A=\frac{\Delta t}{\Delta x^2}\lambda_{i-1,n-1};\quad C=\frac{\Delta t}{\Delta x^2}\lambda_{i,n-1};$$

$$B=0.5\left[\rho_{i,n-1}+\Delta t c_{i,n-1}+(\lambda_{i,n-1}+\lambda_{i-1,n-1})\frac{\Delta t}{\Delta x^2}\right].$$

We have the estimates

$$A>0;\quad C>0;\quad B>\frac{A+C}{2}+\delta;\ \delta>0. \tag{9}$$

Then, according to (3), V_{in} and W_{in} satisfy the conditions

$$\max_i|V_i|\leqslant\max\left\{\frac{|q_1^{(n)}-q_1^{(n-1)}|}{\Delta t},\ \frac{|q_2^{(n)}-q_2^{(n-1)}|}{\Delta t}\right\};$$

$$\max_i|W_i|\leqslant\max\left\{|q_1^{(n-1)}|,\ |q_2^{(n-1)}|,\ \frac{\max\limits_{i=1,\ldots,N-1}|\rho_{i,n-1}u_{i,n-1}|}{\rho_{i,n-1}+\Delta t c_{i,n+1}}\right\}\leqslant\max_i|u_{i,n-1}|.$$

From here, on the n-th layer, by virtue of (6) and the obtained estimates, we have

$$\max_i|u_{in}|\leqslant\Delta t\max\left\{\frac{|q_1^{(n)}-q_1^{(n-1)}|}{\Delta t},\ \frac{|q_2^{(n)}-q_2^{(n-1)}|}{\Delta t}\right\}+\max_i|u_{i,n-1}|$$

$$\leqslant\Delta t\max\left\{\frac{|q_1^{(n)}-q_1^{(n-1)}|}{\Delta t},\ \frac{|q_2^{(n)}-q_2^{(n-1)}|}{\Delta t}\right\}+\Delta t\max\left\{\frac{|q_1^{(n-1)}-q_1^{(n-2)}|}{\Delta t},\ \frac{|q_2^{(n-1)}-q_2^{(n-2)}|}{\Delta t}\right\}$$

$$+\max_i|u_{i,n-2}|\leqslant\ldots\leqslant n\Delta t\max_{1\leqslant k\leqslant n}\left\{\frac{|q_1^{(k)}-q_1^{(k-1)}|}{\Delta t},\ \frac{|q_2^{(k)}-q_2^{(k-1)}|}{\Delta t}\right\}+\max_i|u_{i0}|.$$

Since $t_n = n\Delta t$, we have

$$\max_{i,n}|u_{in}|\leqslant t_n\max_{1\leqslant k\leqslant n}\left\{\left|\frac{dq_1^{(k)}}{dt}\right|,\ \left|\frac{dq_2^{(k)}}{dt}\right|\right\}+\max_i|u_{i0}|,$$

i.e., the solution is bounded and the scheme (5) is stable.

2. We consider the problem

$$\rho(u)\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left[\lambda(u)\frac{\partial u}{\partial x}\right]-c(x,\ t,\ u)u,$$
$$\left.\frac{\partial u}{\partial x}\right|_{x=0}=-q_1(t,\ u|_{x=0});\quad \left.\frac{\partial u}{\partial x}\right|_{x=l}=-q_2(t,\ u|_{x=l});\ u(x,\ 0)=\varphi(x). \tag{10}$$

We make use of the following approximation of the problem (10):

$$\rho_{i,n-1}\frac{u_{in}-u_{i,n-1}}{\Delta t}=\frac{\lambda_{i,n-1}u_{i+1,n}-(\lambda_{i,n-1}+\lambda_{i-1,n-1})u_{in}+\lambda_{i-1,n-1}u_{i-1,n}}{\Delta x^2}-c_{i,n-1}u_{in};$$
$$\frac{u_{1n}-u_{0n}}{\Delta x}=-q_1(u_{0n});\quad \frac{u_{Nn}-u_{N-1,n}}{\Delta x}=-q_2(u_{N,n});\ u_{i0}=\varphi_i. \tag{11}$$

We bring (11) to the form

$$\begin{aligned}
&u_{0n}-u_{1n}=\Delta x q_1(u_{0n}),\\
&a_{10}u_{0n}+a_{11}u_{1n}+a_{12}u_{2n}=m_1,\\
&\ldots\ldots\ldots\ldots\ldots\\
&a_{N-1,\,N-2}u_{N-2,n}+a_{N-1,\,N-1}u_{N-1,n}+a_{N-1,n}u_{Nn}=m_{N-1},\\
&u_{N-1,n}-u_{N,n}=\Delta x q_2(u_{N,n}).
\end{aligned} \tag{12}$$

Here

$$a_{ii} = -\left[\rho_{i,n-1} + (\lambda_{i,n-1} + \lambda_{i-1,n-1})\frac{\Delta t}{\Delta x^2} + \Delta t c_{i,n-1}\right];$$

$$a_{i,i+1} = a_{i+1,i} = \frac{\Delta t}{\Delta x^2}\lambda_{i,n-1};\quad m_i = -\rho_{i,n-1}u_{i,n-1}.$$

We solve the system (12) by an iterative method. We select an initial approximation in the following manner:

$$u_{0n}^{(0)} = u_{0,n-1};\quad u_{N,n}^{(0)} = u_{N,n-1}.$$

Substituting these values in the 2nd, . . . , (N−1)-th equations of the system (12), we obtain $u_{1,n}^{(0)}$, . . . , $u_{N-1,n}^{(0)}$. We bring (12) to a form which is convenient for iteration. We add the first equation to the second one, divided by a_{11}, and the last equation to the next to the last, divided by $a_{N-1,\,N-1}$. We obtain the following system:

$$u_{0n} = \frac{\lambda_{1,n-1}\frac{\Delta t}{\Delta x^2}u_{2,n}}{\rho_{i,n-1} + \lambda_{1,n+1}\frac{\Delta t}{\Delta x^2} + \Delta t c_{1,n-1}} + \Delta x\frac{\rho_{1,n-1} + (\lambda_{1,n-1} + \lambda_{0,n-1})\frac{\Delta t}{\Delta x^2} + \Delta t c_{1,n-1}}{\rho_{1,n-1} + \lambda_{1,n-1}\frac{\Delta t}{\Delta x^2} + \Delta t c_{1,n-1}}q_1(u_{0n})$$

$$-\frac{\rho_{1,n-1}u_{1,n-1}\left[\rho_{1,n-1} + (\lambda_{1,n-1} + \lambda_{0,n-1})\frac{\Delta t}{\Delta x^2} + \Delta t c_{1,n-1}\right]}{\rho_{1,n-1} + \lambda_{1,n-1}\frac{\Delta t}{\Delta x^2} + \Delta t c_{1,n-1}} \equiv \psi_0(u_{0,n}, \ldots, u_{N,n}),$$

$$u_{kn} = -\frac{a_{k-1,k-2}}{a_{k-1,k-1}}u_{k-2,n} - \frac{a_{k-1,k}}{a_{k-1,k-1}}u_{k-1,n} + \frac{m_k}{a_{k-1,k-1}} \equiv \psi_k(u_{0n}, \ldots, u_{Nn}),$$

. . . .

$$u_{Nn} = \frac{\lambda_{N-2,n-1}\frac{\Delta t}{\Delta x^2}u_{N-2,n}}{\rho_{N-1,\,n-1} + \lambda_{N-2,\,n-1}\frac{\Delta t}{\Delta x^2} + \Delta t c_{N-1,\,n-1}}$$

$$-\Delta x\frac{\rho_{N-1,\,n-1} + (\lambda_{N-1,\,n-1} + \lambda_{N-2,\,n-1})\frac{\Delta t}{\Delta x^2} + \Delta t c_{N-1,\,n-1}}{\rho_{N-1,\,n-1} + \lambda_{N-2,\,n-1}\frac{\Delta t}{\Delta x^2} + \Delta t c_{N-1,\,n-1}}q_2(u_{N,n})$$

$$+\frac{\rho_{N-1,\,n-1}u_{N-1,\,n-1}\left[\rho_{N-1,n-1} + (\lambda_{N-1,n-1} + \lambda_{N-2,\,n-1})\frac{\Delta t}{\Delta x^2} + \Delta t c_{N-1,n-1}\right]}{\rho_{N-1,\,n-1} + \lambda_{N-2,\,n-1}\frac{\Delta t}{\Delta x^2} + \Delta t c_{N-1,\,n-1}} \equiv \psi_N(u_{0n}, \ldots, u_{Nn}).$$

For the convergence of the iteration process it is sufficient that

$$\max_i \sum_k \left|\frac{\partial\psi_i}{\partial u_k}\right| < 1.$$

The inequality holds for i = 2, . . . , N−1. For i = 1, N the sum can be made smaller than one at the expense of the choice of Δx. Thus, for sufficiently small Δx, the iteration process converges.

We investigate the stability of the scheme (11). The error in the solution of problem (12) depends on the accuracy with which the values of $u_{0,n}$ and $u_{N,n}$ can be found, since all the remaining u_{in} can be expressed exactly in terms of u_{0n} and u_{Nn} from the 2nd, . . . , (N−1)-th equations of the system (12).

Let u_{0n}^* and u_{Nn}^* be the exact solutions of the system (12). Assume that as a result of the computations one has obtained the values

$$u_{0n} = u_{0n}^* + \alpha_n;\quad u_{Nn} = u_{Nn}^* + \beta_n.$$

Then the system (11) can be written in the form

$$\lambda_{i-1,n-1}\frac{\Delta t}{\Delta x^2}u_{i-1,n}-\left[\rho_{i,n-1}+(\lambda_{i,n-1}+\lambda_{i-1,n-1})\frac{\Delta t}{\Delta x^2}+\Delta t c_{i,n-1}\right]u_{in}+\lambda_{i,n-1}\frac{\Delta t}{\Delta x^2}u_{i+1,n}=-\rho_{i,n-1}u_{i,n-1};\quad u_{i,0}=\varphi_i;\quad u_{0,n}=u^*_{0,n}+\alpha_n;\ u_{N,n}=u^*_{N,n}+\beta_n. \tag{13}$$

Let $\tilde{u}_{i,n}$ be the solution of the problem (13). Let

$$\varepsilon_{i,n}=u_{i,n}-\tilde{u}_{i,n}. \tag{14}$$

Let us prove that $\varepsilon_{i,n}$ remains bounded as $\Delta x\to 0$, i.e., it does not depend on the grid. We insert (14) into (13). We obtain

$$\lambda_{i-1,n-1}\frac{\Delta t}{\Delta x^2}\varepsilon_{i-1,n}-\left[\rho_{i,n-1}+(\lambda_{i,n-1}+\lambda_{i-1,n-1})\frac{\Delta t}{\Delta x^2}+\Delta t c_{i,n-1}\right]\varepsilon_{i,n}+\lambda_{i,n-1}\frac{\Delta t}{\Delta x^2}\varepsilon_{i+1,n}=-\rho_{i,n-1}\varepsilon_{i,n-1}; \tag{15}$$

$$\varepsilon_{i,0}=0;\quad \varepsilon_{0,n}=\alpha_n;\quad \varepsilon_{N,n}=\beta_n.$$

For such a problem we obtain the estimate

$$\max|\varepsilon_{in}|\leqslant n\Delta t\max\left\{\frac{|\alpha_n-\alpha_{n-1}|}{\Delta t},\ \frac{|\beta_n-\beta_{n-1}|}{\Delta t}\right\}. \tag{16}$$

The order of the errors of α_n and β_n are arbitrary. Let

$$|\alpha_n-\alpha_{n-1}|=\frac{k_1}{n^a};\ |\beta_n-\beta_{n-1}|=\frac{k_2}{n^a};\ k_1,\ k_2=\text{const},$$

then

$$\max_{i,n}|\varepsilon_{in}|\leqslant\frac{1}{n^{a-1}}\max\{k_1,\ k_2\}.$$

It is clear that for $a = 1$ $|\varepsilon_{in}|$ remains bounded, while for $a > 1$ $|\varepsilon_{in}|\to 0$. Consequently, the scheme (11) is stable.

3. We consider the problem

$$0<x<c_0\qquad \rho_1(u_1)\frac{\partial u_1}{\partial t}=\frac{\partial}{\partial x}\left[\lambda_1(u_1)\frac{\partial u_1}{\partial x}\right]; \tag{17}$$

$$c_0<x<l\qquad \rho_2(u_2)\frac{\partial u_2}{\partial t}=\frac{\partial}{\partial x}\left[\lambda_2(u_2)\frac{\partial u_2}{\partial x}\right]; \tag{18}$$

$$0<x<c_0\qquad u_1(x,\ 0)=\psi_1(x); \tag{19}$$

$$c_0<x<l\qquad u_2(x,\ 0)=\psi_2(x); \tag{20}$$

$$u_1(0,\ t)=\varphi_1(t);\quad u_2(l,\ t)=\varphi_2(t);\quad u_1(c_0,\ t)=u_2(c_0,\ t); \tag{21}$$

$$\lambda_1(u_1)\frac{\partial u_1}{\partial x}\bigg|_{x=c_0}=\lambda_2(u_2)\frac{\partial u_2}{\partial x}\bigg|_{x=c_0} \tag{22}$$

We divide the interval [0, 1] in such a way that the point $x = c_0$ be one of the approximation nodes. Following (4), we integrate within the limits $[x_i-(\Delta x/2),\ x_i]$, $[x_i,\ x_i+(\Delta x/2)]$ the equation

$$\rho(u)\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left[\lambda(u)\frac{\partial u}{\partial x}\right],$$

where

$$\rho(u)=\begin{cases}\rho_1(u) & 0<x<c_0,\\ \rho_2(u) & c_0<x<l;\end{cases}\qquad \lambda(u)=\begin{cases}\lambda_1(u_1) & 0<x<c_0,\\ \lambda_2(u_2) & c_0<x<l.\end{cases}$$

Adding up the results of the integration, taking into account (21) and (22), we obtain

$$\frac{\rho_{i+0,n-1}+\rho_{i-0,n-1}}{2}\ \frac{u_{in}-u_{i,n-1}}{\Delta t}=\frac{u_{i-1,n}\lambda_{i-\frac{1}{2},n-1}-\left(\lambda_{i-\frac{1}{2},n-1}+\lambda_{i+\frac{1}{2},n-1}\right)u_{in}+\lambda_{i+\frac{1}{2},n-1}u_{i+1,n}}{\Delta x^2}. \tag{23}$$

Expanding both sides of Eq. (23) into Taylor series, we obtain

$$\rho(u)u'_t + o(\Delta t) = \lambda'_x u'_x + \lambda u''_{x^2} + o(\Delta x^2).$$

Consequently, the difference equation (23) approximates the problem (17) with the order $o(\Delta t + \Delta x^2)$.

We have the problem

$$\lambda_{i-1/2,\, n-1}\frac{\Delta t}{\Delta x^2}u_{i-1,n} - \left[\frac{\rho_{i+0,n-1}+\rho_{i-0,n-1}}{2} + \frac{\Delta t}{\Delta x^2}(\lambda_{i-1/2,\, n-1} + \lambda_{i+1/2,\, n-1})\right]u_{in}$$

$$+ \lambda_{i+1/2,\, n-1}\frac{\Delta t}{\Delta x^2}u_{i+1,n} = -\frac{\rho_{i+0,n-1}+\rho_{i-0,n-1}}{2}u_{i,n-1}; \quad (24)$$

$$u_{i0} = \frac{1}{2}(\psi_{i+0} + \psi_{i-0}); \quad u_{0n} = \varphi_{1n}; \quad u_{Nn} = \varphi_{2n}.$$

We obtain a problem of type (5). The scheme (24) is stable if $d\varphi_1/dt$ and $d\varphi_2/dt$ are bounded. In the case of nonlinear boundary conditions

$$u'_x|_{x=0} = -q_1(t,\, u|_{x=0}); \quad u'_x|_{x=l} = -q_2(t,\, u|_{x=l})$$

the approximation of the form (23) leads to a problem of type (11), whose stability has been already studied.

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